Problem 1

$\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{3}\right)^n$ is a geometric series with $r = \frac{x-2}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$

(a)

$\left|\frac{x-2}{3}\right| < 1 \Leftrightarrow -1 < \frac{x-2}{3} < 1 \Leftrightarrow -3 < x-2 < 3 \Leftrightarrow -1 < x < 5$. In that case, the sum of the series is

$\frac{a}{1-r} = \frac{1}{1-\frac{x-2}{3}} = \frac{1}{\frac{3-(x-2)}{3}} = \frac{3}{5-x}$. (b)

Problem 2

$$f(x) = \frac{1+x}{1-x} = (1+x)\left(\frac{1}{1-x}\right) = (1+x)\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} = 1 + \sum_{n=1}^{\infty} x^n + \sum_{n=1}^{\infty} x^n = 1 + 2\sum_{n=1}^{\infty} x^n.$$

Problem 3

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0\cos\omega_0 t \quad (1)$$

$$X_p(t) = A(\omega_0)\cos(\omega_0 t) + B(\omega_0)\sin(\omega_0 t)$$

Substitute in (1) $\Rightarrow$

$$m(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + c(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + k(A\cos\omega_0 t + B\sin\omega_0 t) =$$

$$F_0\cos\omega_0 t \Rightarrow$$

$$[(k-m\omega_0^2)A + cB\omega_0]\cos\omega_0 t + [(k-m\omega_0^2)B - cA\omega_0]\sin\omega_0 t = F_0\cos\omega_0 t \quad (2)$$

$$(2) \Rightarrow \quad (k-m\omega_0^2)A + cB\omega_0 = F_0 \quad (3)$$

$$(k-m\omega_0^2)B - cA\omega_0 = 0 \quad (4)$$

$$(4) \Rightarrow \quad B = A\frac{c\omega_0}{k-m\omega_0^2} \quad (5)$$

Substitute (5) into (3) $\Rightarrow$

$$A = \frac{F_0(k-m\omega_0^2)}{(k-m\omega_0^2)^2 + (c\omega_0)^2} \quad (6)$$

$$\rightarrow B = \frac{F_0 c\omega_0}{(k-m\omega_0^2)^2 + (c\omega_0)^2} \quad (7)$$

The particular solution can be written as $X_p(t) = \tilde{A}(\omega_0)\cos(\omega_0 t + \delta)$

with
$$\tilde{A}(\omega) = \sqrt{A^2+B^2} = \frac{F_0}{\sqrt{(k-m\omega_0^2)^2 + (c\omega_0)^2}}$$

$$\tan\delta = -\frac{B}{A} = -\frac{c\omega_0}{k-m\omega_0^2} \Rightarrow$$

$$\delta = \tan^{-1}\left\{\frac{c\omega_0}{m\omega_0^2 - k}\right\}$$

Problem 4

Definition of Fourier Transform:

$$F(k) = F_x[f(x)] = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi kx} dx \qquad \delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx} dx$$

We have shown in class that:

$$\mathcal{F}_x[\sin(2\pi k_0 x)](k) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\left(\frac{e^{2\pi i k_0 x} - e^{-2\pi i k_0 x}}{2i}\right) dx = \frac{1}{2} i \int_{-\infty}^{\infty}\left[-e^{-2\pi i (k-k_0)x} + e^{-2\pi i (k+k_0)x}\right] dx$$

$$= \frac{1}{2} i [\delta(k+k_0) - \delta(k-k_0)],$$

For sin(4πkₒx) simply replace k_o with $2k_o$ and you have the Fourier transform:

$$F(k) = F_x[\sin[4\pi k_o x]] = \frac{i}{2}\{\delta(k+2k_o) - \delta(k-2k_o)\}$$

Problem 5

Find the solution* of the differential equation

$$y'' + m y = f(x), \quad m \text{ is integer}, \quad f(x) = -f(-x) \text{ [odd function]}$$

Under the conditions $y(0) = y(x=L) = 0$

* Sine solution.

$$y(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (1)$$

Take $f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right), \quad C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$y''(x) = \sum_{n=1}^{\infty}\left(-\left(\frac{n\pi}{L}\right)^2\right) b_n \sin\left(\frac{n\pi x}{L}\right) \quad (2)$$

substitution into $y'' + m y = f(x)$ gives

$$\sum_{n=1}^{\infty}\left[-b_n\left(\frac{n\pi}{L}\right)^2\right]\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} m b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\sum_{n=1}^{\infty}\left\{b_n\left[m - \left(\frac{n\pi}{L}\right)^2\right] - C_n\right\}\sin\left(\frac{n\pi x}{L}\right) = 0 \quad \forall x \in [0, L]$$

$$\Rightarrow b_n\left[m - \left(\frac{n\pi}{L}\right)^2\right] - C_n = 0 \Rightarrow$$

$$b_n = \frac{C_n}{m - \left(\frac{n\pi}{L}\right)^2} \quad \left(m \neq \left(\frac{n\pi}{L}\right)^2\right)$$

$$\boxed{y(x) = \sum_{n=1}^{\infty} \frac{C_n}{m - \left(\frac{n\pi}{L}\right)^2}\sin\left(\frac{n\pi x}{L}\right)}$$

Set: k=m, L=4

Problem 6

(a)

The solution is determined by the separation of variables (the Fourier method):

$$u(x,t) = F(x)G(t).$$

Then

$$\frac{\partial u}{\partial t} = FG', \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

Substituting this into one-dimensional heat equation and separating variables,

$$FG' = c^2 F''G$$

$$\frac{G'}{c^2 G} = \frac{F''}{F} = const = -p^2$$

we obtain the differential equations for $G(t)$ and $F(x)$

$$G' + c^2 p^2 G = 0,$$

$$F'' + p^2 F = 0.$$

Satisfy the boundary conditions:

$$u(0,t) = F(0)G(t) = 0, \qquad u(L,t) = F(L)G(t) = 0, \quad t \geq 0.$$

Thus,

$$F(0) = 0, \qquad F(L) = 0.$$

The general solution for F is

$$F = A\cos px + B\sin px.$$

and

$$F(0) = 0: \qquad A = 0; \qquad F(L) = 0: \qquad B\sin pL = 0$$

which yields

$$\sin pL = 0 \quad (B \neq 0)$$

$$pL = n\pi, \quad p = p_n = \frac{n\pi}{L} \quad (n = 1, 2, \ldots).$$

$$F = F_n = \sin p_n x = \sin \frac{n\pi}{L} x \quad (n = 1, 2, \ldots).$$

The equation for G becomes

$$G' + \lambda_n^2 G = 0, \quad \lambda_n = \frac{cn\pi}{L}.$$

The general solution of this equation is

$$G(t) = G_n(t) = B_n e^{-\lambda_n^2 t} \quad (n = 1, 2, \ldots).$$

Hence the solutions of

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L$$

satisfying

$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t \geq 0.$$

are

$$u_n(x,t) = F_n(x)G_n(t) = B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x \quad (n = 1, 2, \ldots).$$

These functions are called **eigenfunctions** and

$$\lambda_n = \frac{cn\pi}{L}$$

are called **eigenvalues**.

Now we can solve the entire problem by setting

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x.$$

Satisfy the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x).$$

Thus,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx, \qquad n = 1, 2, \ldots.$$

(b) Satisfy the initial conditions:

$$U(x,0)=50\sin(\pi x/40)+30\sin(3\pi x/40)$$

Only the terms $\lambda_1=c\pi/40$ and $\lambda_3=3c\pi/40$ give non-zero contribution to the series solution with $B_1=50$ and $B_3=30$

So the solution is:

$$U(x,t)=50\exp[-\lambda_1^2 t]\sin(\pi x/40)+30\exp[-\lambda_3^2 t]\sin(3\pi x/40)$$